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Magnetized Tori around Kerr Black Holes: Analytic Solutions with a Toroidal Magnetic Field

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ABSTRACT

The dynamics of accretion discs around galactic and extragalactic black holes may be influenced by their magnetic field. In this paper we generalise the fully relativistic theory of stationary axisymmetric tori in Kerr metric of Abramowicz et al.(1978) by including strong toroidal magnetic field and construct analytic solutions for barotropic tori with constant angular momentum. This development is particularly important for the general relativistic computational magnetohydrodynamics that suffers from the lack of exact analytic solutions that are needed to test computer codes.

Key words: black hole physics – accretion discs – MHD – methods:analytical – methods:numerical

1 INTRODUCTION

Accretion discs of black holes have been a subject of intensive observational and theoretical studies for decades. In most of the studies the discs are treated as purely fluid flows (Shakura & Sunyaev 1974; Fishbone & Moncrief 1976; Abramowicz et al. 1978; Rees et al. 1982). That is their magnetic fields are considered as dynamically weak and not important for the force balance determining the disc structure. On the other, it has been long recognised that MHD-turbulence may hold the key to the nature of the disc viscosity that enables the inflow of matter into the black hole (Shakura & Sunyaev 1974). This expectation was confirmed with the discovery of the magneto-rotational instability (Balbus & Hawley 1991) that has become now one of the major topics in theoretical and numerical studies of accretion discs. In the simulations of radiatively inefficient discs they remain relatively weakly magnetized with the magnetization parameter $\beta =$ (gas pressure)/(magnetic pressure) $\simeq 10 - 100$ within the main body of the disc and rising up to $\beta \simeq 1$ only near its inner edge and in the plunging region (Hirose et al. 2004; McKinney & Gammie 2004). However, cooling accretion flows may well evolve towards the state with $\beta < 1$ where the vertical balance against the gravitational force of the central object is achieved by means of magnetic pressure alone (Machida et al. 2006; Pariev et al. 2003).

On the other hand, there is a general consensus that dynamically strong, ordered, poloidal magnetic field is required both for acceleration and collimation of the relativistic jets that are often produced by the astrophysical black hole-accretion disc systems. Such magnetic field is a key ingredient both in the models where the jet is powered by the black hole (Blandford & Znajek 1977) and in the models where it is powered by the accretion disc (Bisnovatyi-Kogan & Ruzmaikin; Blandford & Payne). The origin of this field is not very clear. The most popular idea is that it is carried into central parts of the disc-hole system by the accreting flow itself and that the net magnetic flux gradually builds up there during the long-term evolution of the system (Bisnovatyi-Kogan & Ruzmaikin; Thorne & Macdonald 1982). This is more or less what is observed in recent numerical simulations of radiatively inefficient discs with initially weak poloidal field, (Hirose et al. 2004; McKinney & Gammie 2004). However, the strength of magnetic field that can be reached in this way is still unclear, e.g. (Livio et al. 1999; Uzdensky & Spruit 2005; Meier 2005; McKinney 2005). The most extreme case, where the disc pressure is dominated by ordered poloidal magnetic field is argued by Meier (2005) as a model for the low/hard state of X-ray binaries. Although much weaker magnetic field is required to explain the observed power of quasar jets within the Blandford-Znajek theory, the pressure of poloidal magnetic field still has to be of the same order as the radiative pressure at the inner edge of the radiatively supported accretion discs (Begelman et al. 1984). Bogovalov and Kel'ner (2005) constructed a model of accretion disc where its angular momentum is carried away by a magnetized wind alone and the inflow of matter and magnetic field is driven entirely by the magnetic torque applied to the disc by the wind. These and other results show that the structure of accretion discs

with dynamically strong ordered magnetic is a matter of significant astrophysical interest.

General steady-state axisymmetric magnetized dicks are described by a very complex equation of the Grad-Shafranov type which does not allow general analytic solutions (Lovelace et al. 1986). However, the equilibrium models with pure azimuthal (toroidal) field are much simpler and can be constructed using the approach developed for unmagnetised discs (Abramowicz et al. 1978; Kozlowski et al. 1978). In fact, the predominant motion in accretion discs is a differential rotation and one would expect the azimuthal magnetic field to dominate in the interior of such discs.

Okada et al.(1989) constructed a particular exact solution for the problem of equilibrium torus with purely azimuthal magnetic field. In their approach the Paczynski & Wiita (1980) potential is utilised to introduce the black hole gravity and the flow dynamics is described by the non-relativistic equations. Although this approach seems to work fine in the case of non-rotating black holes, it is no longer satisfactory in the astrophysically important case of rapidly rotating black hole where full relativistic treatment is required. In this paper we develop the general relativistic theory of magnetized tori around rotating black holes. Our analysis closely follows the one of Kozlowski et al.(1978) and Abramowicz et al.(1978), including the notation.

Even if tori with dynamically strong magnetic fields do not exist in nature the exact solutions presented here can still be useful for numerical general relativistic magnetohydrodynamics, GRMHD, which has attracted a lot of interest recently (Koide et al.1999; Komissarov 2001; De Villiers & Hawley 2003; Gammie et al. 2003; Komissarov 2004; Duez et al. 2005; Anton et al. 2006). One of the problems with general relativistic computational magnetohydrodynamics is the lack of exact analytic and semi-analytic solutions that could be used to verify computer codes. The current list of such solutions includes 1) the spherical accretion onto a nonrotating black hole with monopole magnetic field where the magnetic field is dynamically passive (Koide et al.1999), 2) the perturbative force-free solution of Blandford-Znajek (1977), where both the pressure and the inertia of matter is neglected and the black hole rotates slowly, and 3) the so-called Gammie flow, that is a one-dimensional solution of the Weber-Davis type that applies only to the equatorial plane of a black hole (Gammie 1999). One may also try to utilize the axisymmetric semi-analytic self-similar solutions constructed recently by Meliani et al.(2005) but those apply only along the symmetry axis. Obviously, the magnetized torus solution is very welcome an addition to this scarce list. In fact, it is the only solution that is not only a) fully multi-dimensional and b) involves dynamically important magnetic field, but c) also applies in the case of a rapidly rotating black hole. In fact, we have already used this solution to test the 2D GRMHD codes described in Komissarov (2001; 2004).

Throughout the paper we use the relativistic units where $c = G = M = 1$ and $(- + ++)$ signature for the space-time geometry. Following Anile (1989) the magnetic field is rescaled in such a way that the factor 4π disappears from the equations of relativistic MHD.

2 BASIC EQUATIONS

The covariant equations of ideal relativistic MHD are

$$\nabla_\alpha T^{\alpha\beta} = 0, \tag{1}$$

$$\nabla_\alpha {}^*F^{\alpha\beta} = 0, \tag{2}$$

$$\nabla_\alpha \rho u^\alpha = 0 \tag{3}$$

where

$$T^{\alpha\beta} = (w + b^2)u^\alpha u^\beta + (p + \frac{1}{2}b^2)g^{\alpha\beta} - b^\alpha b^\beta \tag{4}$$

is the energy-momentum tensor, w, p and u^α are the fluid enthalpy, pressure and 4-velocity of plasma respectively, and $g_{\alpha\beta}$ is the metric tensor (Dixon 1978; Anile 1989).

$${}^*F^{\alpha\beta} = b^\alpha u^\beta - b^\beta u^\alpha \tag{5}$$

is the Faraday tensor and b^α is the 4-vector of magnetic field. In the fluid frame $b^\alpha = (0, \boldsymbol{B})$, where $\boldsymbol{B}$ is the the usual 3-vector of magnetic field as measured in this frame, and thus

$$u^\alpha b_\alpha = 0. \tag{6}$$

In the following we assume that

(i) the space-time is described by the Kerr metric and that $\{t, \phi, r, \theta\}$ are either Kerr-Schild or Boyer-Lindquist coordinates, so that

$$g_{\mu\nu,t} = g_{\mu\nu,\phi} = 0; \tag{7}$$

(ii) the flow is both stationary and axisymmetric, so that

$$f_{,t} = f_{,\phi} = 0 \tag{8}$$

for any physical parameter f;

(iii) the flow is a pure rotation around the black hole, that is

$$u^r = u^\theta = 0; \tag{9}$$

(iv) the magnetic field is purely azimuthal, that is

$$b^r = b^\theta = 0. \tag{10}$$

In terms of partial derivatives the continuity equation reads

$$(\sqrt{-g}\rho u^\nu)_{,\nu} = 0,$$

where g is the determinant of the metric tensor. Given the symmetry conditions (7) and (8) this equation reduces to

$$(\sqrt{-g}\rho u^i)_{,i} = 0,$$

where here and throughout the whole paper $i = r, \theta$. Finally, the condition (9) tells us that this equation is always satisfied.

Since the Faraday tensor is antisymmetric one may reduce the Faraday equation to

$$(\sqrt{-g}\,{}^*F^{\mu\nu})_{,\nu} = 0.$$

It is easy to see that this equation is also automatically satisfied given the conditions (7-10).

Thus, the only non-trivial results follow from the energy-momentum equation (1). Contracting this equation with the projection tensor $h^\alpha{}_\beta = \delta^\alpha{}_\beta + u^\alpha u_\beta$ we obtain

$$(w + b^2)u_\nu u^\nu{}_{,i} + (p + b^2)_{,i} - b_\nu b^\nu{}_{,i} = 0. \tag{11}$$

In terms of the angular velocity,

$$\Omega = u^\phi / u^t, \tag{12}$$

and the specific angular momentum

$$l = -u_\phi / u_t. \tag{13}$$

(both u_t and u_ϕ are constants of geodesic motion) this equation reads

$$(\ln |u_t|)_{,i} - \frac{\Omega}{1 - l\Omega} l_{,i} + \frac{p_{,i}}{w} + \frac{(\mathcal{L}b^2)_{,i}}{2\mathcal{L}w} = 0, \tag{14}$$

where

$$\mathcal{L}(r, \theta) = g_{t\phi} g_{t\phi} - g_{tt} g_{\phi\phi}. \tag{15}$$

When $b^2 \to 0$ equation (14) reduces to equation 7 in Abramowicz et al.(1978) which describes equilibrium non-magnetic tori. Thus, the Lorentz force vanishes and we have a force-free magnetic torus provided

$$\mathcal{L}b^2 = \text{const}.$$

Far away from the black hole $g_{t\phi} \to 0$, $g_{\phi\phi} \to r^2 \sin^2\theta$, $g_{tt} \to -1$ and this equation reduces to the familiar

$$B^{\hat{\phi}} = \frac{\text{const}}{r \sin\theta}.$$

3 INTEGRABILITY CONDITIONS

For a barotropic equation of state, where

$$w = w(p), \tag{16}$$

equation (14) leads to

$$d\left(\ln |u_t| + \int_0^p \frac{dp}{w} \right) = \frac{\Omega}{1 - l\Omega} dl - \frac{d(\mathcal{L}b^2)}{2\mathcal{L}w} \tag{17}$$

In the case of a non-magnetic torus this equations implies that

$$\Omega = \Omega(l) \tag{18}$$

and thus the surfaces of equal Ω, l, p, and ρ coincide (Abramowicz 1971; Abramowicz et al. 1978). Obviously, this does not have to be the case for magnetized tori. If, however, we still assume that $\Omega = \Omega(l)$ then eq.(17) can be written as

$$d\left(\ln |u_t| + \int_0^p \frac{dp}{w} - \int_0^l \frac{\Omega dl}{1 - l\Omega} \right) = -\frac{d(\mathcal{L}b^2)}{2\mathcal{L}w} \tag{19}$$

that implies that the expression of the right-hand side is a total differential. Hence,

$$\tilde{w} = \tilde{w}(\tilde{p}_m), \tag{20}$$

where $\tilde{w} = \mathcal{L}w$ and $\tilde{p}_m = \mathcal{L}p_m$, where $p_m = b^2/2$ is the magnetic pressure, and eq.(19) integrates to give

$$\ln |u_t| + \int_0^p \frac{dp}{w} - \int_0^l \frac{\Omega dl}{1 - l\Omega} + \int_0^{\tilde{p}_m} \frac{d\tilde{p}_m}{\tilde{w}} = \text{const} \tag{21}$$

Assuming that on the surface of the disc, and hence on its inner edge,

$$p = p_m = 0, \quad u_t = u_{t_{in}}, \quad l = l_{in} \tag{22}$$

one finds the constant of integration as

$$\text{const} = \ln |u_{t_{in}}| - \int_0^{l_{in}} \frac{\Omega dl}{1 - l\Omega}. \tag{23}$$

Following, Abramowicz et al.(1978) we introduce the total potential, W, via

$$W = \ln |u_t| + \int_l^{l_\infty} \frac{\Omega dl}{1 - l\Omega}, \tag{24}$$

where l_∞ is the angular momentum at infinity. Provided that l_∞ is finite we have $u_{t_\infty} = -1$ and $W_\infty = 0$. Using the total potential we can rewrite eq.(21) as

$$W - W_{in} + \int_0^p \frac{dp}{w} + \int_0^{\tilde{p}_m} \frac{d\tilde{p}_m}{\tilde{w}} = 0. \tag{25}$$

With exception for the last term this equation is the same as eq.9 in Abramowicz et al.(1978) and eq.24 in Kozlowski et al.(1978).

4 BAROTROPIC TORI WITH CONSTANT ANGULAR MOMENTUM

4.1 Theory

Here we adopt particular relationships $w = w(p)$, $\Omega = \Omega(l)$, and $\tilde{w} = \tilde{w}(\tilde{p}_m)$ that allow to express the integrals of equation (25) in terms of elementary functions. Namely, we assume that

$$l = l_0, \tag{26}$$

$$p = K w^\kappa \tag{27}$$

$$\tilde{p}_m = K_m \tilde{w}^\eta. \tag{28}$$

The last equation can also be written as

$$p_m = K_m \mathcal{L}^{\eta - 1} w^\eta. \tag{29}$$

Then eq.(25) reduces to

$$W - W_{in} + \frac{\kappa}{\kappa - 1} \frac{p}{w} + \frac{\eta}{\eta - 1} \frac{p_m}{w} = 0, \tag{30}$$

where

$$W = \ln |u_t|. \tag{31}$$

The obvious parameters of the model are κ, η, l_0, and W_{in}. Two more parameters are needed. We chose these to be the enthalpy, w_c, and the magnetization parameter $\beta_c = (p/p_m)_c$ at the disc centre, $r = r_c$. Following Abramowicz et al.(1978) we define the disc centre as one of the two points in the equatorial plane where l_0 equals to the Keplerian angular momentum

$$l_k = \frac{\pm(r^2 \mp 2ar^{1/2} + a^2)}{r^{3/2} - 2r^{1/2} \pm a}, \tag{32}$$

where the upper sign is used if $l_0 > 0$ and the lower sign otherwise (Bardeen et al. 1972). The second point is to the

disc “cusp”, $r_{cusp} < r_c$ (Abramowicz et al. 1978). There exist a number of obvious constraints on the values of these parameters.

In order to avoid divergence of the second and/or the fourth terms in (30) at the disc surface one should have

$$\kappa, \beta > 1.$$

Under this condition the disc surface is fully determined by the choice of W_{in} and does not depend on the disc magnetization. This property has already been noticed in Okada et al.(1989).

Only if

$$|l_0| > |l_{ms}|,$$

where l_{ms} is the radius of the marginally stable Keplerian orbit (Bardeen et al. 1972), the disc is detached from the event horizon (Abramowicz et al. (1978). Solutions attached to the event horizon are improper because they diverge at the event horizon and cannot be continued through it.) The value of l_0 determines the total potential which can be written as

$$W(r,\theta) = \frac{1}{2}\ln\left|\frac{\mathcal{L}}{\mathcal{A}}\right|, \qquad (33)$$

where

$$\mathcal{L} = g_{t\phi}g_{t\phi} - g_{tt}g_{\phi\phi},$$

and

$$\mathcal{A} = g_{\phi\phi} + 2l_0 g_{t\phi} + l_0^2 g_{tt}$$

(see eq.A10).

If $|l_0| \geq |l_{mb}|$ then the disc has finite outer radius only for

$$W_{in} < 0.$$

If $|l_{ms}| < |l_0| < |l_{mb}|$ where l_{mb} is the angular momentum of the marginally bound Keplerian orbit (Bardeen et al. 1972), then the disc remains detached from the black hole only if

$$W_{in} \leq W_{cusp},$$

where W_{cusp} is the value of the total potential at the cusp (Abramowicz et al. 1978).

From eq.(30) one finds the gas pressure

$$p_c = w_c(W_{in} - W_c)\left(\frac{\kappa}{\kappa-1} + \frac{\eta}{\eta-1}\frac{1}{\beta_c}\right)^{-1}, \qquad (34)$$

and then the magnetic pressure

$$p_{m_c} = p_c/\beta_c \qquad (35)$$

at the disc centre. Using these one finds the constants K and K_m of the barotropics (27,29).

Now one can compute the solution at any location inside the disc. Given the coordinates (r,θ) one computes the potential W using equation (33). If $W < W_{in}$ then this point is inside the disc. The next step is to find the enthalpy, w, as a solution of eq.(30). Once w is found one can find p and p_m from eqs.(27) and (29). The 4-velocity vector, $u^\nu = (u^t, u^\phi, 0, 0)$, is given by

$$u^t = -\frac{1}{u_t(1 - l_0\Omega)}, \quad u^\phi = \Omega u^t,$$

Model	l_0	r_{cusp}	r_c	W_{cusp}	W_c	W_{in}	β_c
A	2.8	1.58	4.62	0.702	-0.103	-0.030	0.1
B	2.6	1.78	3.40	-0.053	-0.136	-0.053	1.0

Table 1. Parameters of the models used for test simulations

where Ω can be found via

$$\Omega = -\frac{g_{t\phi} + g_{tt}l_0}{g_{\phi\phi} + g_{t\phi}l_0}$$

(see Appendix.) The 4-vector of magnetic field, $b^\nu = (b^t, b^\phi, 0, 0)$, is given by

$$b^\phi = \pm\sqrt{2p_m/\mathcal{A}}, \quad b^t = l_0 b^\phi$$

(see eq.A16).

4.2 Simulations

In order to illustrate the usefulness of this solution for testing GRMHD computer codes we constructed two equilibrium models and used them to setup the initial solution for 2D axisymmetric simulations using the code described in Komissarov (2004). In both models the black hole has specific angular momentum $a = 0.9$ which gives $r_{mb} = 1.73$, $r_{ms} = 2.32$, $l_{mb} = 2.63$, $l_{ms} = 2.49$. In both cases, we used the same value for the barotropic powers $\kappa, \eta = 4/3$ and the polytropic equation of state with the same ratio of specific heats, $\gamma = 4/3$. The other parameters are given in Table 1. Notice, that $l_0 > l_{mb}$ in model A, whereas in model B one has $l_{ms} < l_0 < l_{mb}$ and this allows accretion through the torus cusp (Abramowicz et al. 1978; Kozlowski et al. 1978).

Initially, the space outside of the tori is filled with a rarefied non-magnetic plasma accreting into the black hole. Its density and pressure are

$$\rho = 10^{-3}\rho_c \exp(-3r/r_c), \quad p = K\rho^\kappa.$$

Its velocity in the frame of local fiducial observer, FIDO, (Thorne & Macdonald 1982) is radial and has the magnitude

$$v = \beta^{\hat{r}}(1 - (r_g/r)^4)$$

where $\beta^{\hat{r}}$ is the radial component of velocity of the spacial grid relative to FIDO. This introduces inflow through the horizon with the local velocity of FIDOs and at the same time allows only small poloidal velocity jump at the torus surface. The latter property allows to avoid strong rarefactions that may originate at the torus surface right at the start of simulations. The initial distributions of ρ, Ω, and β for the model A, that is a more extreme case, are shown in figure 1. Model B appears very similar.

In these simulations we utilized the Kerr-Schild coordinates (r,θ). The computational domain is $[1.35, 53.3] \times [0, \pi]$ with 320 cells in each direction. At lower resolution the effects of numerical diffusion become rather noticeable. The grid is uniform in the θ-direction and the cell size in the radial direction is such that in the equatorial plane $g_{\theta\theta}\Delta\theta^2 = g_{rr}\Delta r^2$. This ensures that computational cells have approximately equal lengths in both directions and throughout the whole grid.

The rotational period of the disc centre is rather long

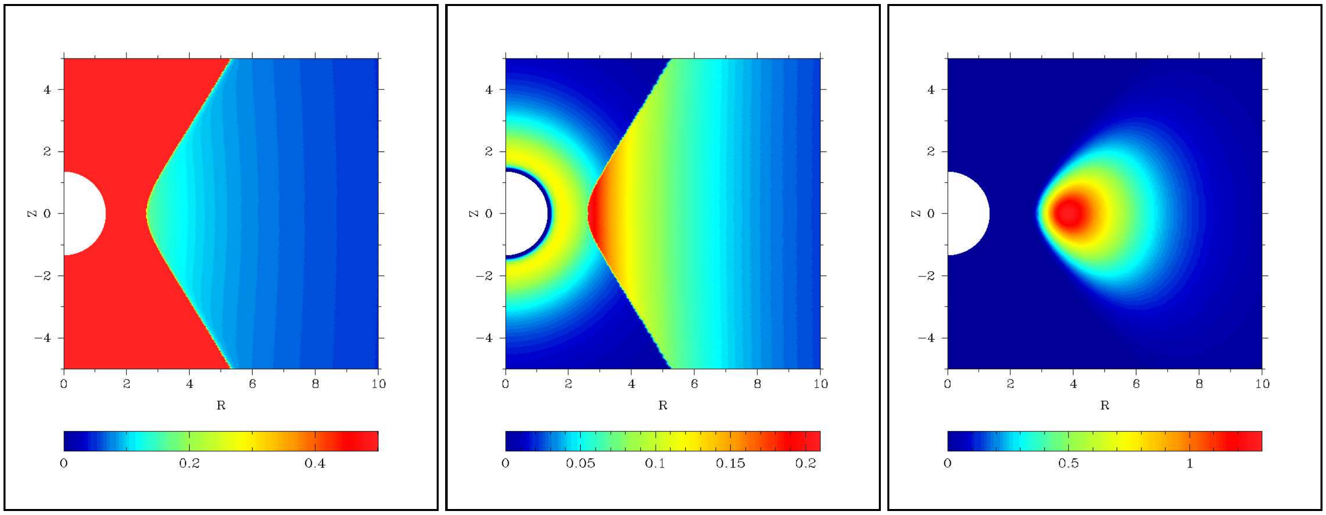

Figure 1. Exact solution for model A. The horizontal coordinate is $R = r \sin\theta$ and the vertical coordinate is $Z = r\cos\theta$ *Left panel:* β, *Middle panel:* Ω, *Right panel:* ρ.

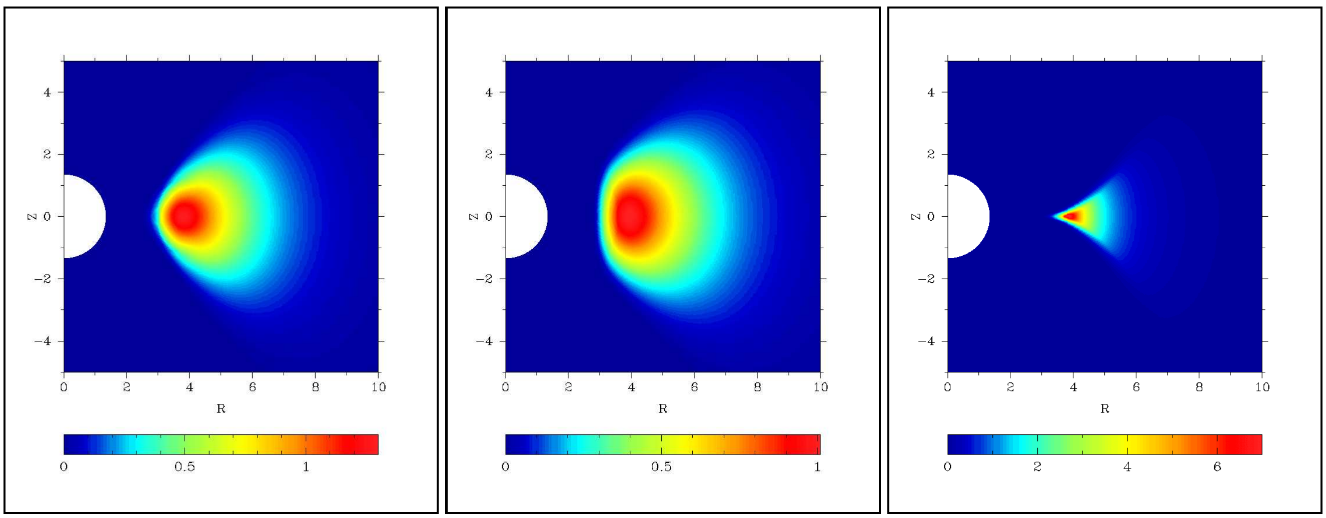

Figure 2. The density distributions for the model A obtained with our actual numerical code and its corrupted versions. In all these plots the linear scaling is used. *Left panel:* the solution obtained with the uncorrupted code at $t = 200$. *Middle panel:* the solution obtained with the corrupted version of the code at $t = 14$. Here we retained only the contribution of magnetic pressure in the Maxwell stress tensor. *Right panel:* the solution obtained with the corrupted version of the code at $t = 14$. Here all the electromagnetic contributions to the stress-energy-momentum tensor were omitted.

$\tau_r = 68$ in model A and $\tau_r = 45$ in model B. However, the dynamical timescale, τ_d, for the disc centre is significantly shorter. Indeed, in both of these models the fast magnetosonic speed in the disc centre is $a_f \simeq 0.18$, whereas the length scale of pressure distribution at this location is $L_c \simeq 1.0$. This gives the dynamical timescale $\tau_d \simeq 6$. Since it is the dynamical timescale that determines how quickly the system reacts to perturbations of its equilibrium state, it is quite sufficient to carry out simulations for only $t =$ few τ_d in order to test the ability of our numerical code to correctly reproduce these equilibrium solutions. In order to demonstrate this we have carried out test simulations for model A not only with our "proper code" but also with its two corrupted versions. In the first version, it is only the electromagnetic pressure, $(B^2 + E^2)/2$, that was taken into account in the calculations of the Maxwell stress tensor. In the second and more drastic version we omitted all the contributions of the electromagnetic field to the stress-energy-momentum tensor. The results are presented in figure 2. One can see that already at $t \simeq 2\tau_d$ the solutions obtained with the corrupted versions are significantly different from the initial equilibrium solution. With the second corrupted version the disc simply collapses towards the equator due to the lack of magnetic support against gravity.

Figure 2 also presents the proper numerical solution at t=200, which is $30 \div 40$ times larger than τ_d. "Naked eye" inspection shows that this solution is very similar to the initial equilibrium one that is presented in figure 1. This is confirmed by the 1D density plots showing in details the distributions along and across the symmetry plane: figure 3 for model A and figure 4 for model B. Other images also reveal what appears to be surface waves propagating away

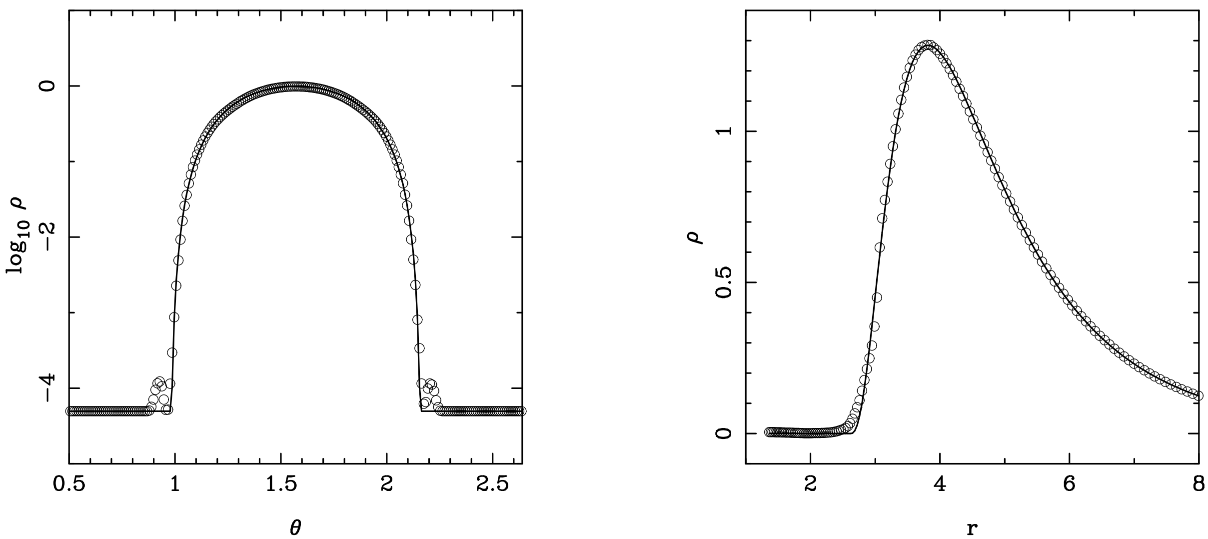

Figure 3. Density distribution at $t = 60$ for model A. The solid lines show the exact equilibrium solution. *Left panel:* $\log_{10}\rho$ against polar angle θ at $r = r_c$, the disc centre location. *Right panel:* ρ against r in the equatorial plane.

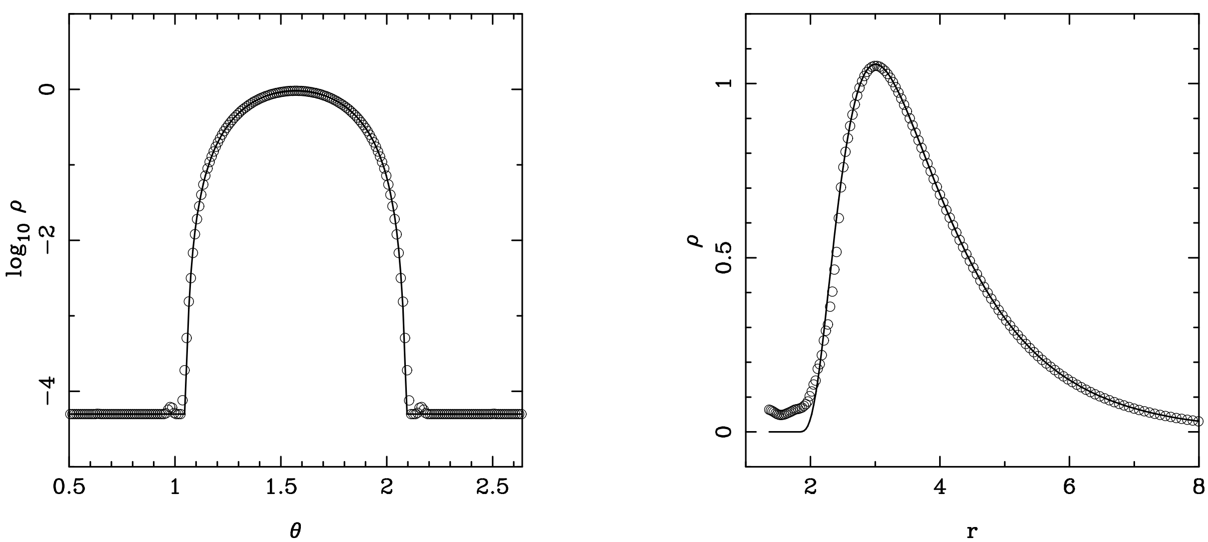

Figure 4. Density distribution at $t = 60$ for model B. The solid lines show the exact equilibrium solution. *Left panel:* $\log_{10}\rho$ against the polar angle θ at $r = r_c$, the disc centre location. *Right panel:* ρ against r in the equatorial plane.

from the black hole and becoming noticeable in the more remote parts of the tori.

5 SUMMARY

In this paper we have generalized the relativistic theory of thick accretion discs around Kerr black holes (Fishbone & Moncrief 1976; Abramowicz et al. 1978; Kozlowski et al. 1978) by including dynamically strong toroidal (azimuthal) magnetic field. As expected, this inclusion of magnetic field leads to a whole new class of equilibrium solutions that differ in strength and spacial distribution of the field. In particular, we have described the way of constructing barotropic tori with constant angular momentum – under such conditions the differential equations of magnetostatics are easily integrated and reduce to simple algebraic equations.

By now it has become clear that magnetic field plays many important roles in the dynamics of astrophysical black hole-accretion disc systems. However, the structure of this field may be more complex than just azimuthal loops and as the result the analytic solutions constructed in this paper may turned out not to be particularly suitable for modelling real astrophysical objects. Even so they will be certainly helpful in testing computer codes for general relativistic MHD that are becoming invaluable tools in the astrophysics of black holes/accretion discs.

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APPENDIX A:

Here we derive equations (11) and (14) of the main paper.

A1 Equation (11)

The contraction of the energy-momentum equation,

$$\nabla_\alpha T^{\alpha\beta} = 0,$$

with the projection tensor

$$h^\alpha{}_\beta = \delta^\alpha{}_\beta + u^\alpha u_\beta$$

leads to

$$\begin{aligned} h^\alpha{}_i \nabla_\gamma \left((w+b^2)u^\gamma u_\alpha\right) &\quad + \\ h^\alpha{}_i \nabla_\gamma \left((p+b^2/2)\delta^\gamma{}_\alpha\right) &\quad + \\ h^\alpha{}_i \nabla_\gamma \left(b^\gamma b_\alpha\right) &\quad = \quad 0, \end{aligned} \tag{A1}$$

where $i = r, \theta$. Applying the product rule and using that $h^\alpha{}_\beta u_\alpha = 0$ and $u^\alpha \nabla_\gamma u_\alpha = 0$ one can write

$$h^\alpha{}_i \nabla_\gamma \left((w+b^2)u^\gamma u_\alpha\right) = (w+b^2)\left[\nabla_\gamma (u^\gamma u_i) - u_i \nabla_\gamma u^\gamma\right].$$

The second term in the square brackets vanishes because of the symmetry conditions (7-10). Applying the well known result for the divergence of a second rank symmetric tensor on obtains

$$\nabla_\gamma (u^\gamma u_i) = \frac{1}{\sqrt{-g}}(\sqrt{-g}u^\gamma u_i)_{,\gamma} - \frac{1}{2} g_{\mu\nu,i} u^\mu u^\nu$$

The first term in this equation vanishes because of the same symmetry conditions whereas

$$g_{\mu\nu,i} u^\mu u^\nu = -2 u_\nu u^\nu{}_{,i}.$$

Thus, one has

$$h^\alpha{}_i \nabla_\gamma \left((w+b^2)u^\gamma u_\alpha\right) = (w+b^2) u_\nu u^\nu{}_{,i}. \tag{A2}$$

Now we deal with the second term in (A1).

$$\begin{aligned} h^\alpha{}_i \nabla_\gamma \left((p+b^2/2)\delta^\gamma{}_\alpha\right) &= h^\gamma{}_i \nabla_\gamma (p+b^2/2) \\ &= (p+b^2/2)_{,i} + u_i u^\gamma (p+b^2/2)_{,\gamma}. \end{aligned}$$

The second term on the right-hand side of this equation vanishes because of the symmetries and hence we have

$$h^\alpha{}_i \nabla_\gamma \left((p+b^2/2)\delta^\gamma{}_\alpha\right) = (p+b^2/2)_{,i}. \tag{A3}$$

As to the third term in (A1),

$$\begin{aligned} h^\alpha{}_i \nabla_\gamma \left(b^\gamma b_\alpha\right) &= h^\alpha{}_i \left(\frac{1}{\sqrt{-g}}(\sqrt{-g} b^\gamma b_\alpha)_{,\gamma} - \frac{1}{2} g_{\mu\nu,\alpha} b^\mu b^\nu\right) \\ &= -\frac{1}{2} h^\alpha{}_i g_{\mu\nu,\alpha} b^\mu b^\nu \\ &= -\frac{1}{2}\left(g_{\mu\nu,i} b^\mu b^\nu + u_i u^\alpha g_{\mu\nu,\alpha} b^\mu b^\nu\right) \\ &= -\frac{1}{2} g_{\mu\nu,i} b^\mu b^\nu \\ &= -\frac{1}{2}(b^2{}_{,i} - 2 b_\nu b^\nu{}_{,i}). \end{aligned} \tag{A4}$$

During this reduction we twice used the symmetry conditions. Substituting of results (A2-A4) into eq.(A1) leads to

$$(w+b^2) u_\nu u^\nu{}_{,i} + (p+b^2)_{,i} - b_\nu b^\nu{}_{,i} = 0, \tag{A5}$$

which is equation (11) of the main paper.

A2 Equation (14)

From the definitions of the angular velocity, Ω, and the angular momentum, l, and the symmetries of the problem it immediately follows that

$$l = -\frac{g_{t\phi} + g_{\phi\phi}\Omega}{g_{tt} + g_{t\phi}\Omega}, \tag{A6}$$

$$\Omega = -\frac{g_{t\phi} + g_{tt} l}{g_{\phi\phi} + g_{t\phi} l}, \tag{A7}$$

where we assumed the Kerr metric in either Kerr-Schild or Boyer-Lindquist coordinates. From the definition of 4-velocity it follows that

$$g_{\mu\nu} u^\mu u^\nu = -1.$$

In the case under consideration this leads to

$$(u^t)^2 = -(g_{tt} + 2 g_{t\phi}\Omega + g_{\phi\phi}\Omega^2)^{-1}, \tag{A8}$$

and

$$u^t u_t = -\frac{1}{1 - l\Omega} \tag{A9}$$

From (A7-A9) one finds

$$(u_t)^2 = \frac{\mathcal{L}}{\mathcal{A}}, \tag{A10}$$

where

$$\mathcal{L} = g_{t\phi} g_{t\phi} - g_{tt} g_{\phi\phi}, \tag{A11}$$

and

$$\mathcal{A} = g_{\phi\phi} + 2 l g_{t\phi} + l^2 g_{tt}. \tag{A12}$$

From the constraint equation (6) and the conditions (9,10) one finds

$$b^t = l b^\phi, \tag{A13}$$

$$b_t = -\Omega b_\phi, \tag{A14}$$

These allow to write

$$b^2 = b_\phi b^\phi (1 - l\Omega), \tag{A15}$$

and

$$b^2 = (b^\phi)^2 \mathcal{A}. \tag{A16}$$

Now we reduce the terms $u_\nu u^\nu{}_{,i}$ and $b_\nu b^\nu{}_{,i}$ in equation (A5). Using the definitions of Ω and l one derives

$$\begin{aligned} u_\nu u^\nu{}_{,i} &= -u^\nu u_{\nu,i} \\ &= -u^t u_{t,i} - u^\phi u_{\phi,i} \\ &= -u^t \left(u_{t,i} - \Omega (l u_t)_{,i}\right) \\ &= -u^t \left(u_{t,i}(1 - l\Omega) - \Omega u_t l_{,i}\right). \end{aligned}$$

The substitution of u^t from (A9) into the last equation gives us

$$u_\nu u^\nu{}_{,i} = (\ln |u_t|)_{,i} - \frac{\Omega}{1 - l\Omega} l_{,i} \tag{A17}$$

Using (A13) and (A14) one writes

$$\begin{aligned} b_\nu b^\nu{}_{,i} &= b_t b^t{}_{,i} + b_\phi b^\phi{}_{,i} \\ &= -\Omega b_\phi (l b^\phi)_{,i} + b_\phi b^\phi{}_{,i} \\ &= b_\phi b^\phi{}_{,i}(1 - l\Omega) - \Omega b_\phi b^\phi l_{,i}. \end{aligned}$$

The substitution of b_ϕ from (A15) into this equation gives

$$b_\nu b^\nu{}_{,i} = b^2(\ln|b^\phi|)_{,i} - \frac{\Omega b^2}{1-l\Omega} l_{,i}. \tag{A18}$$

Substituting (A17,A18) into (A5) one finds

$$w\left((\ln|u_t|)_{,i} - \frac{\Omega}{1-l\Omega} l_{,i}\right) + p_{,i} + b^2\left(\ln\left|\frac{u_t}{b^\phi}\right|\right)_{,i} + b^2_{,i} = 0 \tag{A19}$$

Using (A10) and (A16) one obtains

$$b^2\left(\ln\left|\frac{u_t}{b^\phi}\right|\right)_{,i} + b^2_{,i} = \frac{(\mathcal{L}b^2)_{,i}}{2\mathcal{L}}.$$

Thus, (A19) can be written as

$$(\ln|u_t|)_{,i} - \frac{\Omega}{1-l\Omega} l_{,i} + \frac{p_{,i}}{w} + \frac{(\mathcal{L}b^2)_{,i}}{2\mathcal{L}w} = 0, \tag{A20}$$

which is equation (14) of the main paper.

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